THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 22, 1902, Vice-President Professor Maxime Bôcher in the chair. The attendance at the morning and afternoon sessions was about thirty-five, including the following thirty members of the Society:

Dr. Grace Andrews, Professor Maxime Bôcher, Professor F. N. Cole, Dr. W. S. Dennett, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Dr. G. B. Germann, Dr. Carl Gundersen, Dr. E. R. Hedrick, Dr. G. W. Hill, Dr. E. V. Huntington, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Professor James Maclay, Mr. H. B. Mitchell, Professor W. F. Osgood, Dr. M. B. Porter, Professor J. K. Rees, Miss I. M. Schottenfels, Professor D. E. Smith, Professor P. F. Smith, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor A. G. Webster, Dr. Ruth G. Wood, Professor R. S. Woodward.

The Council announced the election of the following persons to membership in the Society: Professor Edward Brand, Howard College, East Lake, Ala.; Mr. David Raymond Curtiss, Harvard University; Miss Alice Bache Gould, Boston, Mass.; Dr. Carl Gundersen, New York City; Mr. A. F. van der Heyden, Middlesbrough High School, England; Dr. Jean de Seguier, S. J., Paris, France; Mr. J. W. Young, Cornell University. Fourteen applications for membership were received.

Professor T. S. Fiske, whose term as member of the Editorial Board of the *Transactions* had expired, was reelected to this position. Dr. Edward Kasner was reëlected Assistant Secretary of the Society. A petition for the authorization of a Pacific Section of the Society, signed by seventeen members representing the Pacific slope, was received and laid over for consideration at the next meeting of the Council. It was decided to hold the next summer meeting of the Society at Evanston, Ill., in the latter part of August or the first part of September.

The following papers were presented at the meeting:

(1) Dr. E. V. HUNTINGTON: "A complete set of postulates for the theory of absolute continuous magnitude."

(2) Dr. E. V. HUNTINGTON: "Complete sets of postulates for the theories of positive integral and positive rational numbers."

(3) Dr. E. V. HUNTINGTON: "Simplified definition of a group."

(4) Dr. M. B. PORTER: "On the arithmetic nature of the zeros of Bessel functions."

(5) Dr. W. B. FITE: "On a property of groups of order p^n, p being a prime."

(6) Professor L. E. DICKSON: "The groups of Steiner in problems of contact (second paper)."

(7) Dr. VIRGIL SNYDER: "On the forms of quintic scrolls."

(8) Mr. PERCIVAL LOWELL: "On the capture of comets by Jupiter."

(9) Mr. H. L. RIETZ: "On primitive groups of odd order."

(10) Professor MAXIME BÔCHER: "On systems of linear differential equations of the first order."

(11) Dr. E. J. WILCZYNSKI: "Covariants of systems of linar differential equations."

(12) Professor JAMES MACLAY: "On some associated surfaces of negative curvature."

(13) Professor E. W. BROWN: "On the small divisors in the lunar theory."

(14) Mr. OTTO DUNKEL: "Some applications of Green's theorem in one dimension."

(15) Mr. J. W. YOUNG: "On a certain group of isomorphisms."

(16) Dr. A. S. GALE: "On the rank, order, and class of algebraic minimum curves."

(17) Mr. W. H. ROEVER: "Brilliant points and loci of brilliant points."

Mr. Rietz was introduced by Dr. W. B. Fite, and Mr. Roever by Professor Osgood. Mr. Lowell's paper was presented to the Society through Dr. Hill, Mr. Dunkel's through Professor Bôcher, and Mr. Young's through Dr. J. I. Hutchinson. In the absence of the authors, Dr. Fite's paper was read by Dr. Snyder, Mr. Lowell's by Dr. Hill, Mr. Dunkel's by Professor Bôcher, and Mr. Young's by Mr. Rietz. The papers of Professor Dickson, Dr. Wilczynski, and Professor Brown were read by title. At the close of the regular programme Professor A. G. Webster gave a résumé of a paper, also read before the American Physical Society, on the motion of a spherical pendulum, illustrating it with photographs and stereopticon views.

Abstracts of the papers are given below.

Dr. Huntington's first paper presents a complete set of postulates or primitive propositions from which the mathematical theory of absolute continuous magnitude (positive real number) can be deduced.* The fundamental concept involved is that of a "rule of combination" in an assemblage. According to such a rule, any two elements a and b of the assemblage, in a given order, determine the third element $a \circ b$. (The same idea is the fundamental concept in the definition of a "group.") The rule of combination here considered is one upon which six fundamental restrictions are imposed, these requirements being expressed in the following postulates:

1. Whatever elements a and b may be ($a = b$ or $a \neq b$), $a \circ b$ is also an element of the assemblage.

2. $a \circ b \neq a$.

3. $(a \circ b) \circ c = a \circ (b \circ c)$ whenever $a \circ b, b \circ c, (a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the assemblage.

4. When $a \neq b$ at least one of the following conditions is satisfied: either 1° there is an element x such that $a = b \circ x$ or 2° there is an element y such that $a \circ y = b$.

5. (Let the notation $a < b$ indicate that an element y exists such that $a \circ y = b$; and let $a \leqq b$ indicate: $a < b$ or $a = b$.) If S is any infinite sequence of elements, a_1, a_2, a_3, $\cdots$, a_x, a_{x+1}, $\cdots$, such that

$$a_1 < a_2 ; a_2 < a_3 ; \cdots ; a_x < a_{x+1} ; \cdots,$$

and if there is an element c such that $a_x < c$ whenever a_x belongs to S, then there is an element A having the following properties: 1° for every element a_x of S $a_x \leqq A$; and 2° if y and A' are such that $y \circ A' = A$ then there is at least one element of S, say $a_{x'}$, for which $A' < a_{x'}$.

6. Whatever element a may be, there are two elements x and y such that $x \circ y = a$.

The object of the work which follows is to show that these six postulates form a "complete set"; that is, to show that they are 1° consistent, 2° sufficient, and 3° independent. By these three terms is meant: 1° there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements; 2° there is essentially only one such assemblage possible; and 3° none of the six postulates is a consequence of the other five. This

* Cf. O. Hölder, "Die Axiome der Quantität," *Leipziger Berichte*, 1901.

being shown, the postulates 1–6 may be said to define essentially a single assemblage, which is called the "system of absolute continuous magnitude." The element $a \circ b$ is then the "sum" of a and b. From another point of view, the propositions 1–6 may be accepted as expressing in precise mathematical form the essential characteristics of "magnitude" in the popular sense of the word; from this point of view they may properly be called the "axioms" of (absolute continuous) magnitude. From either point of view the propositions 1–6 form a complete logical basis for a deductive mathematical theory. The paper closes with a statement of some unsolved problems concerning possible modifications of the postulates.

By properly modifying the above set of postulates Dr. Huntington constructs in his second paper two complete sets of postulates such that every assemblage which satisfies either of these new sets will be equivalent to the system of positive integers. The first of these sets contains six postulates and the second four. By a further modification, a complete set of five postulates for the theory of positive rational numbers is obtained.

Dr. Huntington's third paper appears in the present number of the BULLETIN.

Except in the case of Bessel functions whose indices are halves of odd integers (in which case the non-zero roots are evidently transcendentally irrational), there seem to be no known theorems concerning the arithmetic nature of the roots of Bessel functions. Dr. Porter applies a theorem of Hurwitz (*Mathematische Annalen*, volume 32) to show that the non-zero roots of all Bessel functions of real rational index are irrationalities of higher order than the second.

If a metabelian group G of order p^m (p being a prime) contains an abelian subgroup of order p^{m-a}, the p^a power of every one of its operators is invariant. Dr. Fite considered the more general case of a group G of order p^m and class k that contains an abelian subgroup of order p^{m-a}. It was shown that in such a group the p^a power of every operator is invariant, if $k \equiv p$. It follows as a corollary that a group of order p^m that contains an abelian subgroup of order p^{m-1} and type $(m_1, m_2, \cdots, m_n)$, where $m_i > 1$ $(i = 1, 2, \cdots, n)$, is either metabelian or of class k, where $k > p$.

Professor Dickson's paper is the continuation of the article with the same title which appeared in the *Transactions* for

January, 1902. Both papers investigate the group G of the equation for the determination of the curves of order $n-3$ having simple contact at $\frac{1}{2}n(n-3)$ points with a given curve C_n of order n having no double points. For $n=4$ these are the 28 bitangents to a quartic curve. The first paper discussed the case n odd; the present paper treats the case n even. The group G is a subgroup of the group of Steiner, the latter being holoedrically isomorphic with the first hypoabelian or the abelian linear group on $2p$ variables with coefficients taken modulo 2, according as n is odd or even. The proofs are simpler, and more direct, than those of Jordan, Traité des Substitutions, pages 229–249.

Several years ago Schwarz found 15 types of quintic scrolls. By means of the dual of the configurations employed by Schwarz, Dr. Snyder extends the subdivision of the forms and finds 28, of which 21 are unicursal, 5 of genus 1, and 2 of genus 2. The most general equation of any type can be written at once when the configuration of nodal lines is given.

Mr. Lowell's paper expresses the action of Jupiter on a comet passing within the sphere of activity of the planet by the equation of energy, and thence by the principle of moving axes deduces Professor H. A. Newton's two important formulæ, extending them to the general case. Tisserand's criterion and Jacobi's integral are shown to be corollaries from the general deduction. The conditions of the relative orbit are then considered and a critical angle found within which, if the comet approaches the planet, the latter's action upon it can never be such as to render its absolute orbit retrograde. All the members of Jupiter's present family are shown to satisfy this criterion, so that the directness of their orbits is, within the limits of the approximation, not only a present but a permanent phenomenon. Lastly the angle of stability is deduced within which Jupiter's control on the capture is complete.

Denoting by G a primitive group of degree n, and by G_1 its maximal subgroup leaving a given letter fixed, the results obtained by Mr. Rietz may be stated as follows: 1° If p^α is the highest power of a prime p which divides the order of G, and if a subgroup P of order p^α is of degree $n-1$, then, unless P is a regular group, or is formed by establishing a simple isomorphism between regular groups, G has an intransitive subgroup of degree n which has a transi-

tive constituent of degree $1 + kp$, and of order lp. 2° If G_1 has an invariant subgroup H of degree $n - \alpha$ ($\alpha > 1$), then G_1 has a transitive constituent whose degree exceeds the degree of any transitive constituent of H. 3° If all the transitive constituents of G_1 are primitive groups of the same degree, G_1 is formed by establishing a simple isomorphism between them. By means of these theorems the determination of the primitive groups of odd order (see BULLETIN, volume 8, page 18) is extended from degree 150 to 243. It results that aside from invariant subgroups of the metacyclic group there are only two such groups within the given limits. They are of degree 169 and of orders 169.7 and 169.21. Each of these groups is solvable. There is then no simple group of odd composite order of degree less than 243.

Professor Bôcher considered the system of differential equations

$$y_i' = \alpha_{i1}y_1 + \cdots + \alpha_{in}y_n + \beta_i \qquad (i = 1, 2, \cdots, n),$$

where the coefficients α_{ij} and β_i are assumed to be functions of the real variable x, which throughout a certain finite interval have at most a finite number of discontinuities, these discontinuities being of such a nature that the integrals

$$\int |\alpha_{ij}|dx, \qquad \int \beta_i dx$$

always converge. The method of successive approximations is used, and it is shown first that for the above system of equations the fundamental existence theorem holds in precisely the same form as it does in the case ordinarily considered where the α's and β's are continuous; and secondly that a small variation in the α's, β's, and the initial values produces a small variation in the solutions.

The covariants considered in Dr. Wilczynski's paper are those of a system of differential equations of the form

$$(1) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0, \end{aligned}$$

under the transformation

$$(2) \quad \xi = \xi(x), \quad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z,$$

where ξ, α, β, γ, δ are arbitrary functions of x. The results obtained are as follows:

An integral rational function of y, z, p_{ik}, q_{ik} and the derivatives of these quantities which is a covariant must be homogeneous in y, z, y', z', etc., and isobaric in all of the quantities entering into it. The assignment of weight is as follows: y and z are of weight zero, p_{ik} of weight one, q_{ik} of weight two; every differentiation increases the weight by unity, and the weight of a product is the sum of the weights of its factors. If $C_{\lambda,w}$ is an integral rational covariant of weight w and degree λ, and if the transformation (2) is made, $C_{\lambda,w}$ will be transformed in accordance with the equation

$$I_{\lambda,w} = \frac{(\alpha\delta - \beta\gamma)^{\frac{\lambda}{2}}}{(\xi')^w} C_{\lambda,w}.$$

Its degree λ is always even.

The system (1) has but three essentially distinct covariants. They are all of the second degree. Putting

$$\rho = 2y' + p_{11}y + p_{12}z, \quad \sigma = 2z' + p_{21}y + p_{22}z,$$

then

$$C_1 = P = z\rho - y\sigma$$

is one of them, of weight 1. There is one of weight 2

$$C_2 = C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz,$$

where the quantities u_{ik} are the same as those so denoted in the author's paper on invariants. Finally there is one of weight 3

$$C_3 = E + 2N,$$

where

$$E = v_{12}z^3 - v_{21}y^3 + (v_{11} - v_{22})yz,$$

$$N = 2(u_{12}z\sigma - u_{21}y\rho) + (u_{11} - u_{22})(z\rho + y\sigma),$$

the quantities v_{ik} also being the same as those so denoted in the author's previous paper on invariants.

In Professor Maclay's paper two related representations of a function of a complex variable by means of surfaces are given, each surface completely representing the function. Some relations between the two surfaces are then exhibited, following which a discussion is made of a class of surfaces

embracing the two above named. All the surfaces under consideration are of negative curvature and have the same spherical representation.

In Professor Brown's paper the difficulties arising in the integration of the equations of motion of the moon from the presence of small divisors are pointed out. The degree of accuracy in terms of higher order is seriously affected thereby. Various methods are developed to avoid, as far as possible, this loss of accuracy, when the author's treatment of the theory is in question. The question as to what degree of accuracy is necessary for the intermediate orbit is fully considered and it is shown that if this orbit be found accurately to m^{10}, all coefficients up to and inclusive of the fourth order can be found to m^8, and within the practical limits set by observation to m^{12-q} for terms of the qth order. If a numerical development be used, it is shown that 10 places of decimals are sufficient for the intermediate orbit.

Mr. Dunkel's paper appears in the present number of the BULLETIN.

The main object of Mr. Young's paper is to determine, by a purely analytic method, the defining relations of the group of isomorphisms I of the non-abelian group of order p^m which contains operators of order p^{m-1}. It is found that I is defined by

$$I^{p^{m-2}(p-1)} = 1,\ J^p = 1,\ K^p = 1,\ I^{-1}JI = J^s,\ I^{p-1}K = KI^{p-1},$$

$$I^{-1}KI = K^t I^{p^{m-2}(p-1)},\ K^{-1}JK = JI^{p^{m-2}(p-1)},$$

where x is any number belonging to the exponent $p-1$ mod p^{m-1}, and where s, t are defined by the relations

$$sx - 1 \equiv 0 \bmod p,\quad 2tx + x - 1 \equiv 0 \bmod p.$$

Dr. Gale considered the equations of a minimum curve in the form

$$x = (1 - s^2)F''(s) + 2sF'(s) - 2F(s),$$

$$y = i(1 + s^2)F''(s) - 2isF'(s) + 2iF(s),$$

$$z = 2sF''(s) - 2F'(s),$$

where $F(s)$ is an algebraic function such that $F'''(s) \neq 0$. Rules were given for the determination of the rank, order, and class of the curve which can be applied if the developments of $F(s)$ in series be known in the vicinity of its poles, branch points, and the point at infinity. Detailed application was made to the case in which $F(s)$ is a two valued function, the case in which $F(s)$ is a rational function having been treated by Lie (*Mathematische Annalen*, volume 14 (1879), pp. 368, 371, 372). Finally attention was called to the bearing of the paper on the theory of algebraic minimum surfaces.

In a polished surface an observer sees images of a source of light. The points in which the surface is pierced by the right lines connecting the eye of the observer with the images are the brilliant points considered in Mr. Roever's paper. When the surface moves, the brilliant points move, and when the motion is very rapid it is possible for the observer to see the locus of the brilliant points. This locus is called a brilliant curve. Such a curve is seen in a carriage wheel with polished spokes when rays of sunlight or rays from a street lamp fall upon it. Closely packed assemblages of brilliant points look like continuous regions of light when the consecutive brilliant points are so close that the eye cannot separate them. A striking example of this is presented by a circular saw which has been polished with emery in a lathe. The consecutive scratches are so close that the corresponding brilliant points cannot be separated by the eye. Each scratch may be considered as being the special position of a variable circular scratch, and hence the isolated brilliant points are points of a brilliant curve. In the present paper are given the analytical conditions which determine the brilliant points of a space curve. The equation of the locus of brilliant points of a one or a two parameter family of curves is obtained. It is shown that both the "carriage wheel curve" and the "saw curve" are represented by equations of the fourth degree, and geometrical constructions are obtained for them. Photographs of the latter curve, in which an arc light is the source and the optical center of the camera lens the recipient, accompany the article. The curves themselves were exhibited by means of very simple apparatus.

EDWARD KASNER,
Assistant Secretary.

NOTE ON THE TRANSFORMATION OF A GROUP INTO ITS CANONICAL FORM.

BY DR. S. E. SLOCUM.

(Read before the American Mathematical Society, December 28, 1901.)

LET

$$X_j \equiv \sum_1^n \xi_{ji}(x_1, \cdots, x_n) \frac{\partial}{\partial x_i} \quad (j = 1, 2, \cdots, r),$$

where the ξ's are analytic functions of n independent variables $x_1, \cdots, x_n$), denote r independent infinitesimal transformations of a given r parameter group. The finite equations of the one-parameter groups generated by each of the infinitesimal transformations X_j $(j = 1, 2, \cdots, r)$ may be obtained by integration of the r simultaneous systems

$$\frac{dx_1'}{\xi_{j1}(x_1', \cdots, x_n')} = \cdots = \frac{dx_n'}{\xi_{jn}(x_1', \cdots, x_n')} = da$$
$$(j = 1, 2, \cdots, r),$$

subject to the condition that $x_i' = x_i$ $(i = 1, 2, \cdots, n)$ for $a = 0$, a being an arbitrary parameter. Let the integrals of these simultananeous systems be represented by the equations

$$x_i' = f_{ik}(x_1, \cdots, x_n, a) \quad (i = 1, 2, \cdots, n\,;\ k = 1, 2, \cdots, r).$$

Performing upon the manifold $x_1, \cdots, x_n$ a general transformation T_1 of the one parameter group generated by X_1 we obtain the manifold $x_1', \cdots, x_n'$; performing upon this latter manifold a general transformation T_2 generated by X_2, we obtain the manifold $x_1'', \cdots, x_n''$, etc. Thus we have

$$x_1' = f_{11}(x_1, \cdots, x_n, a_1), \qquad \cdots x_n' = f_{n1}(x_1, \cdots, x_n, a_1),$$
$$x_1'' = f_{12}(x_1', \cdots, x_n', a_2), \qquad \cdots x_n'' = f_{n2}(x_1', \cdots, x_n', a_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_1^{(r)} = f_{1r}(x_1^{(r-1)} \cdots, x_n^{(r-1)}, a_r), \cdots x_n^{(r)} = f_{nr}(x_1^{(r-1)}, \cdots, x_n^{(r-1)}, a_r),$$

where $a_1, \cdots, a_r$ are arbitrary parameters. Eliminating $x_1', \cdots, x_n^{(r-1)}$ between these equations, we have

$$x_i^{(r)} = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n),$$

or, if we denote the transformed variables $x_1^{(r)}, \cdots, x_n^{(r)}$ by $x_1', \cdots, x_n'$ respectively,

$$x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n), \tag{1}$$

in which the parameters $a_1, \cdots, a_r$ are all essential. These equations define ∞^r transformations of the given r-parameter group.*

This method will now be applied to finding the finite equations of the group whose infinitesimal transformations are

$$X_1 \equiv \frac{\partial}{\partial x_2}, \quad X_2 \equiv x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.$$

The infinitesimal transformation $\dfrac{\partial}{\partial x_2}$ generates the one-parameter group whose finite equations are

$$x_1' = x_1, \quad x_2' = x_2 + a_1, \tag{2}$$

and similarly $x_2 \dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_1}$ generates the one-parameter group whose finite equations are

$$x_1'' = x_1' + a_2, \quad x_2'' = x_2' e^{a_2}. \tag{3}$$

Eliminating x_1', x_2' between equations (2) and (3) we have

$$x_1'' = x_1 + a_2, \quad x_2'' = x_2 e^{a_2} + a_1 e^{a_2},$$

or, replacing x_1'', x_2'' by x_1', x_2'.

$$x_1' = x_1 + a_2, \quad x_2' = x_2 e^{a_2} + a_1 e^{a_2}. \tag{4}$$

These equations define a transformation T_a of the given group G. Similarly, the equations defining a transformation T_b of G are

$$x_1'' = x_1' + b_2, \quad x_2'' = x_2' e^{b_2} + b_1 e^{b_2}. \tag{5}$$

The transformation $T_b T_a$, obtained by the composition of the transformations T_a and T_b in the order named, is defined by the equations

$$x_1'' = x_1 + a_2 + b_2, \quad x_2'' = x_2 e^{a_2+b_2} + a_1 e^{a_2+b_2} + b_1 e^{b_2}. \tag{6}$$

* Lie. Continuierliche Gruppen, pp. 192–197.

If this is equivalent to a transformation T_c of G, we also have

$$x_1'' = x_1 + c_2, \quad x_2'' = x_2 e^{c_2} + c_1 e^{c_2}. \tag{7}$$

Therefore

$$\begin{aligned} c_1 &= a_1 + b_1 e^{-a_2} \equiv \varphi_1(a, b), \\ c_2 &= a_2 + b_2 \quad\quad \equiv \varphi_2(a, b). \end{aligned} \tag{8}$$

The c's are finite for every finite system of values of the a's and b's, but the transformation $T_a T_b$, or T_c, may not be generated by an infinitesimal transformation of the group, as will appear later.

In order to transform the finite equations

$$x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \quad (i = 1, 2, \cdots, n) \tag{1}$$

of the group into their canonical form, Lie proceeds as follows.* By differentiation and elimination we obtain the differential equations

$$\frac{\partial x_i'}{\partial a_k} = \sum_1^r {}_j \psi_{jk}(a_1, \cdots, a_r) \xi_{ji}(x_1', \cdots, x_n')$$

$$(i = 1, 2, \cdots, n\,;\ k = 1, 2, \cdots, r),$$

which are satisfied identically by equations (1) above. Since the determinant of the $\psi_{jk} \not\equiv 0$, these equations may be written in the form

$$\xi_{ji}(x_1', \cdots, x_n') = \sum_1^r {}_k a_{jk}(a_1, \cdots, a_r) \frac{\partial x_i'}{\partial a_k}$$

$$(i = 1, 2, \cdots, n\,;\ j = 1, 2, \cdots, r),$$

where the determinant of the $a_{jk}(a) \not\equiv 0$, and no linear relation of the form $e_1 \xi_{1i}(x') + \cdots + e_r \xi_{ri}(x') \equiv 0$, with constant coefficients e persists simultaneously for $i = 1, 2, \cdots, n$. The canonical form of the finite equations of the group can now be obtained by integration of the simultaneous system

$$\frac{dx_1'}{\sum_1^r {}_j \lambda_j \xi_{j1}(x')} = \cdots = \frac{dx_n'}{\sum_1^r {}_j \lambda_j \xi_{jn}(x')} = dt,$$

* Transformationsgruppen, vol. 3, pp. 609–611.

that is to say, by integration of the simultaneous system

$$\frac{dx_i'}{dt} = \sum_{k,j}^{1\cdots r} \lambda_j \alpha_{jk}(a_1, \cdots, a_r) \frac{\partial x_i'}{\partial a_k} \quad (i = 1, 2, \cdots, n),$$

subject to the condition that $x_i' = x_i (i = 1, 2, \cdots, n)$ for $t = 0$. Consequently $a_1, \cdots, a_r$, considered as functions of t, satisfy the simultaneous system

$$\frac{da_k}{dt} = \sum_{1}^{r} \lambda_j \alpha_{jk}(a_1, \cdots, a_r) \qquad (k = 1, 2, \cdots, r). \tag{9}$$

Since $x_i' = x_i (i = 1, 2, \cdots, n)$ for $t = 0$, $a_1, \cdots, a_r$ must assume the values $a_1^0, \cdots, a_r^0$ for $t = 0$, where $a_1^0, \cdots, a_r^0$ are the parameters which furnish the identical transformation. Integrating the simultaneous system (9) subject to the condition that $a_k = a_k^0 (k = 1, 2, \cdots, r)$ for $t = 0$, we obtain the integrals

$$a_k = M_k(\lambda_1 t, \cdots, \lambda_r t) \qquad (k = 1, 2, \cdots, r),$$

or, if we denote the parameters $\lambda_1 t, \cdots, \lambda_r t$ by $\mu_1, \cdots, \mu_r$, respectively, we have

$$a_k = M_k(\mu_1, \cdots, \mu_r) \qquad (k = 1, 2, \cdots, r).$$

Inserting these values of $a_1, \cdots, a_r$ in equations (1), we have

$$x_i' = f_i(x_1, \cdots, x_n, M_1(\mu), \cdots, M_r(\mu)) \qquad (i = 1, 2, \cdots, n),$$

which are the *canonical* equations of the given group.

Consider the application of this method to the finite equations (4) on page 281. Since the equations

$$\begin{aligned} x_1' &= x_1 + a_2 \qquad \equiv f_1(x, a), \\ x_2' &= x_2 e^{a_2} + a_1 e^{a_2} \equiv f_2(x, a) \end{aligned} \tag{4}$$

define ∞^2 transformations T_a which constitute a group, functional equations persist of the form

$$f_i(f(x, a), b) = f_i(x, \varphi(a, b)) \qquad (i = 1, 2),$$

or

$$f_i(x', b) = f_i(x, c) \qquad (i = 1, 2). \tag{10}$$

The functions φ_1, φ_2 in equations (8) are independent of one another with respect to b_1, b_2, for

$$\frac{\partial \varphi_k}{\partial b_j} = \begin{vmatrix} e^{-a_2}, & 0 \\ 0, & 1 \end{vmatrix}$$

is not identically zero. Therefore we may regard x_1, x_2, a_1, a_2, c_1, c_2 as independent variables, and x_1', x_2', x_1'', x_2'', b_1, b_2 as dependent variables. Thus the differentiation of the functional equations (10), that is, of

$$x_1' + b_2 = x_1 + c_2,$$

$$x_2' e^{b_2} + b_1 e^{b_2} = x_2 e^{c_2} + c_1 e^{c_2},$$

with respect to the a's gives

$$\frac{\partial x_1'}{\partial a_1} + \frac{\partial b_2}{\partial a_1} = 0, \qquad \frac{\partial x_1'}{\partial a_2} + \frac{\partial b_2}{\partial a_2} = 0,$$

(11)

$$e^{b_2}\frac{\partial x_2'}{\partial a_1} + (x_2' + b_1)e^{b_2}\frac{\partial b_2}{\partial a_1} + e^{b_2}\frac{\partial b_1}{\partial a_1} = 0,$$

$$e^{b_2}\frac{\partial x_2'}{\partial a_2} + (x_2' + b_1)e^{b_2}\frac{\partial b_2}{\partial a_2} + e^{b_2}\frac{\partial b_1}{\partial a_2} = 0.$$

In order to obtain expressions for $\frac{\partial b_j}{\partial a_k}$, we differentiate equations (8) with respect to a_1, a_2, and thus obtain

$$0 = \frac{\partial b_2}{\partial a_1}, \qquad 0 = 1 + \frac{\partial b_2}{\partial a_2}, \qquad 0 = 1 + e^{-b_2}\frac{\partial b_1}{\partial a_1},$$

$$0 = -b_1 e^{-b_2} + e^{-b_2}\frac{\partial b_1}{\partial a_2},$$

whence

$$\frac{\partial b_2}{\partial a_1} = -e^{b_2}, \quad \frac{\partial b_1}{\partial a_2} = b_1, \quad \frac{\partial b_2}{\partial a_1} = 0, \quad \frac{\partial b_2}{\partial a_2} = -1.$$

Inserting these values, equations (11) become

$$(12) \qquad \frac{\partial x_1'}{\partial a_1} = 0, \quad \frac{\partial x_1'}{\partial a_2} = 1, \quad \frac{\partial x_2'}{\partial a_1} = e^{b_2}, \quad \frac{\partial x_2'}{\partial a_2} = x_2'.$$

These equations are of the form

$$\frac{\partial x_i'}{\partial a_k} = \Psi_{1k}(a, b)\Phi_{1i}(x', b) + \Psi_{2k}(a, b)\Phi_{2i}(x', b)$$
$$(i = 1, 2;\ k = 1, 2),$$

where

$$\Psi_{ik}(a, b) \equiv \frac{\partial b_i}{\partial a_k} \quad (i = 1, 2\,;\ k = 1, 2).$$

Therefore

$$\Psi_{11} = -e^{a_2}, \quad \Psi_{12} = b_1, \quad \Psi_{21} = 0, \quad \Psi_{22} = -1,$$

and, consequently,

$$\Phi_{11} = 0, \quad \Phi_{12} = -1, \quad \Phi_{21} = -1, \quad \Phi_{22} = -x_2' - b_1.$$

If we insert in equations (12) the values of x_1', x_2' derived from

$$x_2' = x_2 e^{a_2} + a_1 e^{a_2}, \qquad x_1' = x_1 + a_2, \tag{4}$$

they become equations between the independent quantities x_1, x_2, a_1, a_2, b_1, b_2, and must therefore be satisfied identically. Hence equations (12) will still persist identically in virtue of equations (4) if we assign definite values $\bar{b}_1$, $\bar{b}_2$ to b_1, b_2. For this purpose let $b_1 = 1$, and denote the functions $\Psi(a, \bar{b})$ and $\Phi(x', \bar{b})$ by $\psi(a)$ and $\xi(x')$ respectively. Then we have

$$\frac{\partial x_i'}{\partial a_k} = \psi_{1k}(a)\xi_{1i}(x') + \psi_{2k}(a)\xi_{2i}(x') \tag{13}$$
$$(i = 1, 2\,;\ k = 1, 2),$$

where

$$\xi_{11} = 0, \quad \xi_{12} = -1, \quad \xi_{21} = -1, \quad \xi_{22} = -x_2' - 1,$$
$$\psi_{11} = -e^{a_2}, \quad \psi_{12} = 1, \quad \psi_{21} = 0, \quad \psi_{22} = -1.$$

The determinant of the ψ's, namely

$$\begin{vmatrix} \psi_{11}, & \psi_{12} \\ \psi_{21}, & \psi_{22} \end{vmatrix} = \begin{vmatrix} -e^{a_2}, & 1 \\ 0, & -1 \end{vmatrix},$$

not being identically zero, equations (13) may be solved for the ξ's, giving

$$\xi_{ji}(x_1', x_2') = \alpha_{j1}(a)\frac{\partial x_i'}{\partial a_1} + \alpha_{j2}(a)\frac{\partial x_i'}{\partial a_2}$$
$$(i = 1, 2\,;\ j = 1, 2),$$

where

$$\alpha_{11} = -e^{-a_2}, \quad \alpha_{12} = 0, \quad \alpha_{12} = -e^{-a_2}, \quad \alpha_{22} = -1.$$

In order to obtain the system of functions $a_k = M_k(\mu_1, \mu_2)$ $(k = 1, 2)$ which, when introduced into the finite equations (4) of the group, will transform these equations into their canonical form, it is necessary to integrate the simultaneous system

$$\frac{da_k}{dt} = \sum_1^2 \lambda_j \alpha_{jk}(a_1, a_2) \qquad (k = 1, 2)$$

subject to the condition that $a_k = a_k^0$ $(k = 1, 2)$ for $t = 0$, that is, we must integrate the simultaneous system

$$\frac{da_1}{dt} = -\lambda_1 e^{-a_2} - \lambda_2 e^{-a_2}, \qquad \frac{da_2}{dt} = -\lambda_2,$$

subject to the condition that $a_1 = a_2 = 0$ for $t = 0$. The integrals of these equations are

$$a_1 = -\frac{\lambda_1 t + \lambda_2 t}{\lambda_2 t}(e^{\lambda_2 t} - 1) \equiv M_1(\lambda_1 t, \lambda_2 t),$$

$$a_2 = -\lambda_2 t \equiv M_2(\lambda_1 t, \lambda_2 t),$$

or, if we let $\mu_1 = -(\lambda_1 + \lambda_2)t$, $\mu_2 = -\lambda_2 t$,

$$a_1 = -\frac{\mu_1}{\mu_2}(e^{-\mu_2} - 1) \equiv M_1(\mu_1, \mu_2), \tag{14}$$

$$a_2 = \mu_2 \equiv M_2(\mu_1, \mu_2).$$

Inserting these values, equations (4) become

$$x_1' = x_1 + \mu_2,$$

$$x_2' = x_2 e^{\mu_2} + \frac{\mu_1}{\mu_2}(e^{\mu_2} - 1),$$

which is the canonical form of the equations defining a transformation T_μ of the group. These equations are of precisely the same form as the equations obtained by summation of the series

$$x_i' = x_i + \sum_1^2 \mu_j X_j x_i + \frac{1}{2!}\sum_1^2 \sum_1^2 \mu_j \mu_k X_j X_k x_i + \cdots \qquad (i = 1, 2),$$

where

$$X_1 \equiv \frac{\partial}{\partial x_2}, \quad X_2 \equiv x_2 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1}.$$

For every finite system of values of the μ's, the a's are finite. Consequently every transformation of the family

T_μ belongs to the family T_a, and is generated by an infinitesimal transformation of the group. Since $M_1(\mu)$, $M_2(\mu)$ in equations (14) are independent functions of the μ's, Lie assumes that every transformation of the family T_a belongs to the family T_μ. But although the functions a_1, a_2, defined by equations (14), are independent of the μ's since the Jacobian

$$\frac{\partial(a_1, a_2)}{\partial(\mu_1, \mu_2)} \equiv \frac{e^{\mu_2} - 1}{\mu_2 e^{\mu_2}}$$

is not identically zero, nevertheless for certain values of the a's the μ's are infinite. Thus, solving equations (14), we have

$$\mu_1 = \frac{a_1 a_2 e^{a_2}}{e^{a_2} - 1} \equiv N_1(a_1, a_2), \qquad \mu_2 = a_2 \equiv N_2(a_1, a_2).$$

For $a_1 \neq 0$ and a_2 an even multiple of $\pi\sqrt{-1}$, μ_1 becomes infinite, and this transformation of the family T_a is distinct from any transformation of the family T_a for which the μ's are finite. But infinite values of the μ's are excluded from consideration, for $\mu_1 = -(\lambda_1 + \lambda_2)t$, $\mu_2 = -\lambda_2 t$, and since t cannot be infinite if μ_1 or μ_2 is infinite, λ_1 or λ_2 must be infinite, and by supposition the λ's are arbitrary but definite constants. Consequently we cannot assume that every transformation of the family T_a belongs to the family T_μ.

This necessitates a restriction upon the criterion for the continuity of a group. For a system of values of the a's for which one or both of the functions $N_1(a)$, $N_2(a)$ are infinite there is no equivalent transformation of the family T_μ, and consequently such a transformation T_a cannot be generated by an infinitesimal transformation of the group. For example, the transformation considered above, for which $a_1 \neq 0$, $a_2 =$ an even multiple of $\pi\sqrt{-1}$, cannot be generated by an infinitesimal transformation of the group. Such a transformation is termed a *singular* transformation, and a group which contains a singular transformation is said to be discontinuous.

A group is said to be continuous if it contains no singular transformation. In other words, an r-parameter group is said to be continuous if the composition of two arbitrary transformations T_a and T_b of the group, generated by the infinitesimal transformations

$$a_1 X_1 + \cdots + a_r X_r, \quad b_1 X_1 + \cdots + b_r X_r$$

respectively, is equivalent to a transformation T_c of the group, generated by the infinitesimal transformation

$$c_1 X_1 + \cdots + c_r X_r,$$

with finite parameters $c_1, \cdots, c_r$; that is to say, if a system of finite values of the c's can be found to satisfy the symbolic equation $T_b\, T_a = T_c$. On page 282 we saw that the composition of the two arbitrary transformations T_a and T_b of the family defined by equations (4) was equivalent to a transformation T_c of the family, with finite parameters c. But equations (4) were not in their canonical form, and therefore it did not necessarily follow that the transformation T_c could be generated by an infinitesimal transformation of the group, as shown above. Consequently, if the finite equations of a group are not in their canonical form, the condition that for every finite system of values of the a's and b's a finite system of the c's can be found to satisfy the symbolic equation $T_b\, T_a = T_c$ is a necessary but not a sufficient condition for the continuity of the group.

UNIVERSITY OF CINCINNATI,
December, 1901.

SOME APPLICATIONS OF GREEN'S THEOREM IN ONE DIMENSION.

BY MR. OTTO DUNKEL.

(Read before the American Mathematical Society, February 22, 1902.)

GREEN'S theorem ordinarily has reference to Laplace's equation in either two or three dimensions. It has been generalized however in the case of two dimensions by replacing Laplace's equation by the general homogeneous linear differential equation of the second order. In the generalized form the theorem relates not only to the given differential equation, but also to its adjoint differential equation.* A further extension of the theorem is possible by considering a differential equation of the nth order in two or more independent variables, and its corresponding adjoint.†

* Cf. Encyklopädie, II, A. 7 c., p. 513.

† Cf. Darboux, Théorie des Surfaces, vol. 2, pp. 72, 74, for the case of two independent variables.

It is desired here to consider the extension of the theorem to differential equations in only one independent variable, *i. e.*, to the ordinary homogeneous linear differential equation of the second and higher orders, and to indicate very briefly some results which can be obtained from this extension, which we may call Green's theorem in one dimension.

We shall consider the general homogeneous linear differential expression

$$(1) \qquad P(y) = \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1} \frac{dy}{dx} + p_n y,$$

in which $p_1, p_2, \cdots, p_n$ are continuous real functions of the real independent variable x in the interval $a \leqq x \leqq b$. We shall further assume that such of their derivatives as may be needed exist and are continuous.

If y and z are any two functions of x with continuous derivatives of orders up to and including the nth, then by integrating the formula of Lagrange,

$$z\,P(y) - y\,\overline{P}(z) = \frac{d}{dx} P(y, z),$$

where $\overline{P}(z)$ and $P(y, z)$ are respectively the adjoint and bilinear differential expressions of $P(y)$,* we shall obtain the following integral formula, which we may designate as Green's theorem in one dimension,

$$(2) \qquad \int_a^b [z\,P(y) - y\,\overline{P}(z)]\,dx = P(y, z)\Big|_a^b$$

$$P(y, z)\Big|_a^b = P[y(b), z(b)] - P[y(a), z(a)].$$

When the expression to which Green's theorem is applied is self-adjoint,† there is a second associated integral formula which here becomes

$$\int_a^b \sum_{i=0}^{n} (-1)^i p_{n-i} y^{(i)} z^{(i)}\,dx$$

$$(3) \qquad = \int_a^b y\,P(z)\,dx - \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j y^{(j)} \frac{d^{i-j-1}}{dx^{i-j-1}} (p_{n-i} z^{(i)})\Big|_a^b$$

$$= \int_a^b z\,P(y)\,dx - \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j z^{(j)} \frac{d^{i-j-1}}{dx^{i-j-1}} (p_{n-i} y^{(i)})\Big|_a^b,$$

* Cf. Schlesinger's Handbuch, vol. 1, pp. 53, 55, or Darboux, Théorie des surfaces, vol. 2, pp. 99, 100.

† Cf. Schlesinger's Handbuch, vol. 1, p. 73; or Darboux, Théorie des surfaces, vol. 2, p. 119, for characteristic form of self-adjoint expressions.

where

$$n = 2m, \quad y^{(i)} = \frac{d^i y}{dx^i}$$

This formula is the analogon of the three term form of Green's theorem in the potential theory

$$\iint \left[\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} \right] dx\, dy$$

$$= - \iint U \Delta V dx\, dy - \int U \frac{\partial V}{\partial n} ds$$

$$= - \iint V \Delta U dx\, dy - \int V \frac{\partial U}{\partial n} ds.*$$

When $n = 2$ the general differential expression

$$(4) \qquad P(y) = \frac{d^2 y}{dx^2} + p \frac{dy}{dx} + q y$$

can always be rendered self-adjoint by multiplying by the non-vanishing factor

$$(5) \qquad K = e^{\int_a^x p dx}$$

and we can then write

$$(6) \qquad K P(y) = \frac{d K y'}{dx} + G y,$$

where

$$G = q K = q e^{\int_a^x p dx}$$

Applying Green's theorem to (6) we have

$$(2') \qquad \int_a^b K [z P(y) - y P(z)] dx = K \left[z \frac{dy}{dx} - y \frac{dz}{dx} \right]_a^b$$

$$(3') \qquad \int_a^b \left[K \frac{dy}{dx} \frac{dz}{dx} - G y z \right] dx = K y \frac{dz}{dx} \Big|_a^b - \int_a^b K y P(z) dx$$

$$= K z \frac{dy}{dx} \Big|_a^b - \int_a^b K z P(y) dx.$$

*Cf. Encyklopädie, II, A. 7 c., for the corresponding three term formula in the case of the general self-adjoint homogeneous partial differential equation of the second order.

Special theorems may be obtained from these last two formulæ by making different suppositions as to y and z. If y and z are solutions of $P(y) = 0$ and $b = x$, we shall have from (2′) Abel's formula

$$z\frac{dy}{dx} - y\frac{dz}{dx} = \frac{C}{K} \qquad (C = \text{constant}). \tag{7}$$

If y is a solution of

$$\frac{dK_1y'}{dx} + G_1y = 0,$$

and z is a solution of

$$\frac{dK_2z'}{dx} + G_2z = 0,$$

we have from (3′)

$$K_1z\frac{dy}{dx} - K_1y\frac{dz}{dx}\Big|_a^b = \int_a^b (G_2 - G_1)yz\,dx - \int_a^b (K_2 - K_1)\frac{dy}{dx}\frac{dz}{dx}\,dx$$

the formula of Sturm;* and we may therefore regard Sturm's work as an application of Green's theorem in one dimension.

The following theorem results from (2′):

If there exists a function z of x such that it and its first two derivatives are continuous in the interval $a \leqq x \leqq b$ and satisfy the relations

$$z > 0, \tag{9}$$

$$P(z) \leqq 0, \tag{10}$$

then $P(y) = 0$ is a non-oscillatory differential equation in the interval $a \leqq x \leqq b$.†

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*Cf. Encyklopädie II, A. 7a, p. 442.

†Cf. a paper by Professor Bôcher, BULLETIN, May, 1901, p. 333, for definition of non-oscillatory differential equation. In the same paper, p. 335, is a different proof of this theorem, and a number of special tests obtained from it.

If we had used in Green's theorem $P(y)$ instead of the self-adjoint expression $KP(y)$, the relation (10) would be replaced by

$$\bar{P}(z) \leqq 0.* \tag{10'}$$

A theorem in regard to the non-homogeneous linear differential equation of the second order

$$P(y) = r \tag{11}$$

can be obtained, if we suppose r is a function of x that does not change sign in the interval $a \leqq x \leqq b$.

Suppose y is a solution of (11) that vanishes together with its first derivative at a; and z is a solution of the reduced equation

$$P(z) = 0$$

that vanishes at a and again at c, $(a < c \leqq b)$, but not between a and c. Then the theorem is that *y will not vanish again in the interval $a < x \leqq c$.*†

The general Green's function in one dimension for equations of the nth order has been defined by Professor Bôcher.‡ As an example of the use of such a function, we shall consider for simplicity only the equation of the second order (4). For a given point ξ within the interval $a \leqq x \leqq b$, there is in general one and only one Green's function $G(x, \xi)$ of the first kind. Applying Green's theorem to $G(x, \xi)$ and any particular solution y of (4), we have

$$K(\xi)\, y(\xi) = - K(x)\, y(x) \frac{\partial G(x, \xi)}{\partial x}\Bigg|_a^b. \tag{12}$$

This corresponds to the integral formula which gives the value of a solution of Laplace's equation at any point within a closed curve by means of the values that the solution takes on along the curve.

For equations of higher order there are corresponding formulæ.

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February, 17, 1902.

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* A theorem corresponding to this for the elliptic type of the homogeneous linear partial differential equation of the second order has recently been given by Professor Bôcher in his lectures on partial differential equations

† This theorem can also be proved by using the formula expressing the solution of (11) by means of a solution of the reduced equation.

‡ BULLETIN, April, 1901, p. 297.

ON THE FORMS OF QUINTIC SCROLLS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, February 22, 1902.)

THE only paper devoted to the systematic study of quintic scrolls is that by Schwarz,* in which surfaces are classified according to the configuration of the double curve. Fifteen types are enumerated, of which ten are unicursal, four of genus 1, and one of genus 2. Schwarz regarded the scroll as generated by the line of intersection of corresponding planes in two developables the sum of whose classes is 5.

It seems desirable to enumerate the types from the dual standpoint, *i. e.*, regarded as generated by the line joining corresponding points of two plane or twisted curves. Of course, the same forms will appear that were obtained before, but the relations between subforms are more clearly brought out.

A. Unicursal Quintics.

Let a line δ and a unicursal quartic c_4 be put in (1, 1) correspondence and lines joining corresponding points be drawn; these lines will generate a quintic scroll having δ for a simple director line; any plane through δ will cut the scroll in 4 generators which intersect in 6 points and each cuts δ in one point. The plane is a fourfold tangent plane; hence the double curve is of order 6 (type II).† The double curve does not cut δ.

Let δ cut c_4, then a generator coincides with δ; the residual curve is of order 5, it cuts δ in 3 points (VI).

Let δ cut c_4 twice, then δ is a triple line; the residual curve is of order 3, it cuts δ in 2 points (III).

Let δ cut c_4 three times; δ counts as a fourfold line and there is no other nodal curve on the surface (I).

Let c_4 be replaced by a double conic c_2^2; the double curve consists of the conic and a twisted quartic which is cut by every generator twice (VIII).

Thus far the correspondence is between a line and a curve of order 4; it remains to consider the correspondence between a conic c_2 and a unicursal cubic c_3. The general case gives nothing new from the standpoint of the double curve.

* "Ueber die geradlinigen Flächen fünften Grades" *Crelle*, vol. 67.

† The Roman numerals refer to types given by Schwarz.

Let c_3 be replaced by a triple line and c_2 by a double line; this accounts for a nodal curve of order 4; a unicursal quintic has a nodal line of order 6, hence 2 double generators must exist (V). This type can be defined by a (2, 3) correspondence between the points of two lines which are skew to each other, the correspondence to have one double element.

Let c_3 be replaced by a triple line; the plane of c_1 will cut three generators from the surface, all of which pass through the trace T of the triple line. Let one generator cut c_2 in P_1 and P_2; if P_1 corresponds to T and P_2 corresponds to T, the generator joining P_1P_2 becomes double and the double curve consists of the triple line, the double generator, and a conic not in the plane of c_2 (IV).

Let c_3 be replaced by a double line; the plane of c_3 contains two generators, which may coincide as in the last case. The double curve consists of the double line, a double generator, and a twisted quartic which cuts the double generator in two points (VII).

Let c_2 be replaced by a double line; in general there is one point from which issue three pairs of generators; let this triple point be resolved, which can always be done by changing the parameters. The double curve is now a quintic with a double point (X).

Finally, let there be a (2, 2) correspondence between two conics which intersect in a double element. As each generator can cut each conic but once, but must cut the complete nodal curve three times, and as the nodal curve is of order 6, it follows that the residual curve is a third conic section (IX).

From this method of generation the equations can be at once written down in the most general form.

Special forms of these types may exist: thus, in (I) δ may be (1) a fourfold directrix, (2) a triple directrix and a single generator, (3) a double directrix and a double generator, (4) a single directrix and a threefold generator; all of these may be obtained in the same way by letting the directing line intersect the directing curve in the proper number of points. (II) may have (5) a line for simplest curve on the surface, or (6) a conic. In (III), the triple line may be (7) triple directrix, (8) a double directrix and single generator, or (9) a single directrix and double generator. Similarly in (IV), we have types (10), (11), (12). The last case presents a surface having a singularity not met with in scrolls of lower order. Let a (1, 2) correspondence exist between the points of a line and a conic

which have one point in common. The line will be simple directrix and double generator. For example, let

$$x = 0, \quad y = 0, \quad z = \frac{1}{\mu} \text{ be the equations of the line,}$$

$$x = \lambda^2, \quad y = \lambda, \quad z = 0 \text{ be the equations of the conic;}$$

and let $\lambda = \mu^3$. The equation of the scroll is

$$y(xw - y^2)^2 = x^3z^3.$$

It is unicursal; every generator cuts one other generator at a point on the conic and two others at the point in which it intersects the line. The triple line $x = 0$, $y = 0$ and the double conic $z = 0$, $y^2 = wz$ make a nodal curve of order 5; hence a double generator exists. But $x = 0$, $y = 0$ is a single directrix; hence but one generator can pass through each point. The directrix line counts for a curve of order 4 as a component of the nodal sextic, though any plane through it will cut two generators from the surface. Any plane section not through a generator will cut the surface in a unicursal quintic curve having a tacnode and a simple branch through it at the trace of the directrix line; hence it can have but two other nodes, namely, at the points in which the cutting plane intersects the double conic. In (V) the directrices may be (13) distinct, or (14) coincident. In (VI) the line may be (15) double directrix, or (16) simple directrix and generator. In (VII) there are no subdivisions (17); nor in (VIII), (18); nor in (IX), (19). In (X) there are two possible types (20), (21).

B. Quintics of Genus 1.

By joining corresponding points of two non-singular cubics which have one self-corresponding point, a quintic scroll of genus 1 is generated. The nodal curve is of order 5 (XI). There are no subdivisions of this form (22).

One cubic may be replaced by a triple line, leaving a conic for residual nodal curve (XII), (23). The double conic may degrade into a double generator and a double directrix line (XIII); the directrices may be (24) distinct or (25) coincident.

A double line can be brought into (1, 1) correspondence with a non-singular cubic; the scroll thus defined has the double line and a twisted quartic for nodal curves (XIV) (26).

C. Quintics of Genus 2.

The cubic may be replaced by a triple line; the nodal curve is completed by one double directrix and one triple directrix (XV); the skew directrices may be (27) distinct or (28) coincident.

CORNELL UNIVERSITY,
January 25, 1902.

SIMPLIFIED DEFINITION OF A GROUP.

BY DR. E. V. HUNTINGTON.

(Read before the American Mathematical Society, February 22, 1902.)

UP to the present time no attempt seems to have been made to prove the independence of the postulates employed to define a group, and as a matter of fact the definition usually given contains several redundancies.* These redundancies are removed in the following note, the number of necessary postulates being reduced to three, and the independence of these three being established.

Fundamental Concepts.

A *class* of objects is determined when any condition is given such that every object in the universe must either satisfy or not satisfy the condition. Every object which satisfies the condition is said to *belong to* the class. (We shall agree to exclude the case of a class to which no element belongs.)

A class thus defined is usually called, in mathematical parlance, an *assemblage* (Menge, ensemble), every object which belongs to the class being called an *element* of the assemblage.

A *rule of combination* in an assemblage is any rule or agreement by which, when any two elements (whether the same or different) are given, in a definite order, some object (which may or may not itself belong to the assemblage †) is uniquely determined.

If the first of the two given elements is denoted by a and the second by b, then the object which they determine is denoted by $a \circ b$ (read: "a with b").

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*See for example H. Weber, Algebra. Vol. II. (1899), pp. 3–4.

†The object determined by any two elements of the assemblage always will belong to the assemblage if postulates 1, 2, 3 are satisfied, as we prove below in 10.

When two different symbols x and y are used to represent the same object, we indicate this fact by the notation $x = y$.

Definition of a Group.

Any assemblage in which the rule of combination denoted by $\circ$ satisfies the three following postulates we shall call a *group* with respect to this rule of combination:

1. *Given any two elements a and b, there is an element x such that $a \circ x = b$.*

2. *Given any two elements a and b, there is an element y such that $y \circ a = b$.*

3. *If a, b, c, $a \circ b$, $b \circ c$, and either $(a \circ b) \circ c$ or $a \circ (b \circ c)$ are elements of the assemblage, then*

$$(a \circ b) \circ c = a \circ (b \circ c).*$$

The usual definition of a group, as given for example in Weber's Algebra, loc. cit., contains not only these three postulates, but also certain others, which we proceed now to deduce as consequences of our postulates 1, 2, 3, thus establishing the equivalence of the two definitions.

4. *Lemma. If $a \circ x = a$ then $b \circ x = b$.* (That is, if an element x, when combined with any particular element a, leaves that element unchanged, then x will have the same property when combined with any other element b.)

Proof: By 2 take y so that $y \circ a = b$; then by hypothesis, $y \circ (a \circ x) = b$. But by 3, $y \circ (a \circ x) = (y \circ a) \circ x$; therefore $b \circ x = b$.

5. *Lemma. If $y \circ a = a$ then $y \circ b = b$.*

Proof: By 1 take x so that $a \circ x = b$; then by hypothesis, $(y \circ a) \circ x = b$. But by 3, $(y \circ a) \circ x = y \circ (a \circ x)$; therefore $y \circ b = b$.

6. *If $a \circ b = a \circ b'$ then $b = b'$.*

Proof: By 1 take x so that $b' \circ x = b$; then by hypothesis, $a \circ (b' \circ x) = a \circ b'$ whence by 3, $(a \circ b') \circ x = (a \circ b')$. Therefore by 4, $b' \circ x = b'$, that is, $b = b'$.

* Postulate 3 does not demand that either of the objects $(a \circ b) \circ c$ and $a \circ (b \circ c)$ shall belong to the assemblage; if either one of them does belong, however, then the other must also.

7. *If* $a \circ b = a' \circ b$ *then* $a = a'$.

Proof: By 2 take y so that $y \circ a' = a$; then by hypothesis, $(y \circ a') \circ b = a' \circ b$, whence by 3, $y \circ (a' \circ b) = (a' \circ b)$. Therefore by 5, $y \circ a' = a'$, that is, $a = a'$.

8. *The element* x *in* 1 *is uniquely determined by* a *and* b (*by* 6).

9. *The element* y *in* 2 *is uniquely determined by* a *and* b (*by* 7).

10. *Whatever elements* a *and* b *may be,* $a \circ b$ *is also an element of the assemblage; that is, there is an element* c *such that* $a \circ b = c$.

Proof: By 1 take e so that $a \circ e = a$, (1°)

and b' so that $b \circ b' = e$; (2°)

then by 2 take c so that $c \circ b' = a$. (3°)

The element c thus determined is the element required in the theorem.

For, by 1 take β so that $a \circ \beta = c$, (4°)

and β' so that $\beta \circ \beta' = e$. (5°)

From (3°) and (4°) we have $(a \circ \beta) \circ b' = a$ and from (1°) and (5°) we have $a \circ (\beta \circ \beta') = a$, whence by 3

$$(a \circ \beta) \circ \beta' = (a \circ \beta) \circ b'.$$

Therefore by 6, $\beta' = b'$.

Then (5°) becomes $\beta \circ b' = e$,

whence by (2°) $\beta \circ b' = b \circ b'$.

Therefore by 7, $\beta = b$; hence by (4°) $a \circ b = c$ as desired.

We can now justify our definition of a group as follows: our postulates 1 and 2, combined with theorems 8 and 9, are equivalent to postulate 4 in Weber's definition, and our postulate 3 to his 2. Weber's postulate 1 appears here as theorem 10, and his postulate 3 is contained in our theorems 6 and 7. Hence the two definitions are strictly equivalent.

A simple example of a group is the assemblage of all integral numbers, positive, negative, and zero, with $a \circ b = a + b$. Another example is the assemblage of positive rational numbers, with $a \circ b = a \times b$.

Independence of Postulates 1, 2, 3.

The independence of the postulates 1, 2, 3 can be readily established by the method now commonly used in such cases.

Thus, the assemblage of positive integers, with the rule of combination $a \circ b = a$, satisfies 2 and 3, but not 1. Hence 1 is not a consequence of 2 and 3.

Again, the assemblage of positive integers, with $a \circ b = b$, satisfies 1 and 3, but not 2. Hence 2 is not a consequence of 1 and 3.

Finally, the assemblage of positive rational fractions, with $a \circ b = a/b$, satisfies 1 and 2, but not 3. Hence 3 is not a consequence of 1 and 2.*

Finite Groups.

A *finite group* is a group which contains only a finite number of elements. If the number of elements is n, the group is said to be of the nth *degree*.

A simple example of a group of the nth degree can be constructed as follows: take the assemblage of the positive integers from 1 to n and let

$$\begin{aligned} a \circ b &= a + b \quad \text{when} \quad a + b \leqq n, \\ &= a + b - n \quad \text{when} \quad a + b > n. \end{aligned}$$

First Definition of a Finite Group.

If we wish to restrict our definition to groups of the nth degree, we may add to the postulates 1, 2, 3 the following:

11. *The assemblage shall contain only n elements.*

The postulates 1, 2, 3, 11 will then be independent of each other when $n > 2$.

Thus, to prove the independence of 1, take the assemblage of the first n positive integers ($n > 1$), with $a \circ b = a$, and to prove the independence of 2 take the same assemblage, with $a \circ b = b$.

To prove the independence of 3, take the assemblage of the first n positive integers ($n > 2$), with the rule of combination defined as follows:

$$\begin{aligned} a \circ b &= a + b - \quad \text{when} \quad a + b \leqq n, \\ &= a + b - n \quad \text{when} \quad a + b > n \end{aligned}$$

except that

$$a \circ b = 2 \quad \text{when} \quad a + b = 1 \text{ or } n + 1,$$

and

$$a \circ b = 1 \quad \text{when} \quad a + b = 2 \text{ or } n + 2.$$

* Since each of the three systems here mentioned satisfies 10, we see that no one of the postulates 1, 2, 3 can be deduced from the other two, even with the aid of 10.

This assemblage does not satisfy 3, since $(1 \circ 1) \circ 2 = 1 \circ 2 = 3$, while $1 \circ (1 \circ 2) = 1 \circ 3 = 4$ when $n > 3$, and $= 2$ when $n = 3$.

To prove the independence of 11, consider any infinite group, such as either of the examples mentioned above.*

It should be noticed that the proof of 10 can be made much simpler when we are able to use 11. Thus, let a be any fixed element, and let c_k run through the whole assemblage. Then for every element c_k there is an element b_k such that $a \circ b_k = c_k$, by 1; and all these b's are different elements, by 6; that is, b_k also runs through the whole assemblage. Hence $a \circ b$ is always an element of the assemblage, for the given element a. Similarly for every other element a.

Second Definition of a Finite Group.

We have just shown that a finite group may be defined by a set of *four* postulates, each independent of the other three. The definition usually given, as for example by Weber, loc. cit., includes *five* postulates, viz., the propositions here numbered 10, 6, 7, 3 and 11.† It is interesting to notice, in conclusion, that these five also form a set of independent postulates when $n > 2$.

Thus, to prove the independence of 10, consider the assemblage of the first n positive integers with $a \circ b = a + b$.

To prove the independence of 6 and 7 consider the assemblage of the first n positive integers ($n > 1$), with $a \circ b = a$ and $a \circ b = b$ respectively.

And to prove the independence of 3 and 11 consider the assemblages already used for the same purpose in the preceding paragraph.

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*Since each of the four systems here considered satisfies 10, we see that no one of the postulates 1, 2, 3, 11 can be deduced from the other three, even with the aid of 10.

†The propositions 1 and 2 being readily deduced from these as theorems.

NOTE ON ISOTROPIC CONGRUENCES.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, December 28, 1901.)

CONSIDER a sphere S of radius unity and center at the origin of coördinates, and a surface S_1 corresponding to S by orthogonality of linear elements. By a theorem of Ribaucour * we know that S_1 is the mean surface of an isotropic congruence C. If S_1 is taken to define an infinitesimal deformation of S, the associate† surface in this deformation will be a minimal surface S_2;‡ then S and S_2 correspond by parallelism of tangent planes at corresponding points. Moreover, if the reciprocal character of the relation existing between these latter two surfaces is noted, and S is considered as the associate in the deformation of S_2, then the surface S_3 corresponding to S_2 with orthogonality of linear elements is the adjoint minimal surface of S_2.

Darboux has shown § that the following relations exist between the cartesian coördinates of these four surfaces:

$$\begin{aligned} x_1 &= x_3 - y_3 z + z_3 y, \\ y_1 &= y_3 - z_3 x + x_3 z, \\ z_1 &= z_3 - x_3 y + y_3 x. \end{aligned} \tag{1}$$

Let S be referred to its asymptotic lines; then S_2 and S_3 will be referred to the double system of lines which is conjugate for each.‖ As the latter are adjoint minimal surfaces, this double system is made up of the lines of length zero on each surface. Weierstrass has shown ** that the coördinates of S_3 can be expressed as the following functions of u and v, parameters referring to the lines of length zero:

$$\begin{aligned} x_3 = &\frac{1-u^2}{2} f''(u) + uf'(u) - f(u) \\ &+ \frac{1-v^2}{2} f_1''(v) + vf_1'(v) - f_1(v), \end{aligned}$$

* Étude des Elassoïdes, chap. 8 (Mémoire couronné par l'Academie de Belgique); Darboux, vol. 4, p. 15.
† Bianchi, p. 279.
‡ Darboux, vol. 4, p. 96.
§ Ibid., p. 66.
‖ Bianchi, p. 284.
** Darboux, vol. 1, p. 289.

$$(2)\qquad \begin{aligned} y_2 &= i\frac{1+u^2}{2}f''(u) - iuf'(u) + if(u) \\ &\quad - i\frac{1+v^2}{2}f_1''(v) + ivf_1'(v) - if_1(v), \\ z_2 &= uf''(u) - f'(u) + vf_1''(v) - f_1'(v), \end{aligned}$$

where accents denote differentiation. Since S_2 is the adjoint minimal surface of S_2, the coördinates x_2, y_2, z_2 have the following expressions:*

$$\begin{aligned} x_2 &= i\left[\left(\frac{1-u^2}{2}f''(u) + uf'(u) - f(u)\right)\right. \\ &\quad \left. - \left(\frac{1-v^2}{2}f_1''(v) + vf_1'(v) - f_1(v)\right)\right], \end{aligned}$$

$$(3)\qquad \begin{aligned} y_2 &= -\left(\frac{1+u^2}{2}f''(u) - uf'(u) + f(u)\right. \\ &\quad \left. + \frac{1+v^2}{2}f_1''(v) - vf_1'(v) + f_1(v)\right), \\ z_2 &= i\left(uf''(u) - f'(u) - vf_1''(v) + f_1'(v)\right). \end{aligned}$$

Since S and S_2 have parallel tangent planes at corresponding points and S is a sphere of radius unity, the coördinates x, y, z are equal to the direction cosines of the normal to S. Hence these coördinates have the following expressions:†

$$(4)\qquad x = \frac{u-v}{1+uv}, \quad y = i\frac{v-u}{1+uv}, \quad z = \frac{uv-1}{1+uv}.$$

Substituting in (1) the above expression for x, ···, z_2, we find

$$(5)\qquad \begin{aligned} x_1 &= \frac{u+v}{1+uv}(f'+f_1') - 2\frac{f+f_1}{1+uv}, \\ y_1 &= i\frac{v-u}{1+uv}(f'+f_1') - 2i\frac{f_1-f}{1+uv}, \\ z_1 &= \frac{uv-1}{1+uv}(f'+f_1') - 2\frac{uf_1+vf}{1+uv}. \end{aligned}$$

But these are the expressions found by Darboux‡ for the

* Darboux, vol. 1, p. 323

† Ibid., p. 296.

‡ Darboux, vol. 4, p. 17.

cartesian coördinates of a point on the mean surface of an isotropic congruence, where the functions f and f_1 which enter in these expressions refer to the minimal surface which is the envelope of the mean planes of the congruence. In the present case these functions refer to S_2, the adjoint minimal surface of S_2, the latter being the associate of S in the deformation determined by S_1. Hence we have the following theorem:

The middle envelope of an isotropic congruence is the adjoint of the minimal surface which is the associate surface in the infinitesimal deformation of the sphere, the directrix of the congruence.

PRINCETON,
November, 1901.

KRONECKER'S LECTURES ON THE THEORY OF NUMBERS.

Vorlesungen über Mathematik, von Leopold Kronecker, herausgegeben unter Mitwirkung einer von der Königlich Preussischen Akademie der Wissenschaften eingesetzten Kommission. In zwei Teilen. Zweiter Teil. *Vorlesungen über Allgemeine Arithmetik.* Bearbeitet und herausgegeben von DR. KURT HENSEL. Erster Abschnitt: *Vorlesungen über Zahlentheorie.* Erster Band. 8vo., xvi + 509 pp. Teubner, 1901.

IN the summer semester of 1841 Lejeune Dirichlet gave, for the first time, a course of lectures under the title of "Zahlentheorie" at the university of Berlin.* These lectures were attended by Kronecker. The subject was soon afterward added to the regular announcements not only at Berlin but also at the other German universities. The fact that during the winter semester of the current university year at least seven of the German universities offered courses on this subject, given by such well-known men as Frobenius, Weber, and Gordan, is sufficient evidence of the abiding interest in the theory which, according to Gauss, excels all other parts of pure mathematics in "its magic charm" and "its inexhaustible richness."

The volume before us is to be followed by a second on the same subject. In these volumes the editor aims to develop the theory of numbers in such a manner as to preserve the personal imprint of Kronecker, but to fill out the lacunæ which the lectures of this great arithmetician naturally pre-

* Under the title of "Anfangsgründe der höheren Arithmetik" Dirichlet offered a course on the theory of numbers at Berlin as early as 1833.

sent, especially since they were prepared for students who were generally familiar with the elements of the subject. In the present volume the editor carefully distinguishes his own contributions in the extensive list of "Anmerkungen" given at the end. The volume is divided into four parts, which bear the following headings: Divisibility and congruences of numbers, Domains of rationality and the theory of modular systems, Application of analysis to problems in the theory of numbers, General theory of residues of powers and proof of the theorem on the arithmetical progression.

The first three lectures are devoted to a historical introduction in which the chief contributions, beginning with the Greeks and ending with Dirichlet, Jacobi and Kummer, are sketched. In these introductory lectures one begins to feel that his guide is remarkably skilful in the choice of methods and takes a special interest in leading his hearers by the most direct routes to the questions which await solution. "Kronecker influenced the mathematical thinking of Germany as much through his lectures as through his published writings. He was a very stimulating and interesting lecturer. To an unusual degree he took his hearers into his confidence and allowed them the privilege of watching the actual evolution of his thoughts."*

Even in the first lecture attention is called to some interesting properties which await proof. In speaking about Euclid's proof of the fact that the number of primes is infinite, Kronecker remarks that it appears as if there were an infinite number of primes which differ from each other by the least possible number, viz., by two units. This property is included in the very difficult question in regard to the intervals in which prime numbers necessarily lie. Euclid's process gives intervals which increase too rapidly with the number. That there are indefinitely large intervals in which no prime number lies follows directly from the fact that the $n-1$ consecutive numbers

$$n!+2, \quad n!+3, \quad n!+4, \quad \cdots, \quad n!+n$$

do not include any prime, since $n!+i$ $(1<i<n+1)$ is divisible by i.

The law of the succession of prime numbers has been the object of many investigations. Among the various memoirs on this difficult subject, that presented in 1850 to the Academy of St. Petersburg by Tchebycheff is especially fa-

* Fine, "Kronecker and his arithmetical theory of the algebraic equation," BULLETIN, vol. 1 (1892), p. 183.

mous, since it proves the very useful theorem that there is always at least one prime between a and $2a - 2$ $(a > \frac{1}{2})$, a truth which had been conjectured by Bertrand at a much earlier date. In volume 4 of the *American Journal of Mathematics*, Sylvester gives some extensions of this memoir, which has been made more accessible through a reprint in Liouville, volume 17 (1852).

The first part of the present volume, Divisibility and congruences of numbers, begins with a study of the number concept. The definition of number which is given by Euclid and his followers is criticised as having no value for the matheematician. No definition may be considered justified until a method is supplied for determining in every concrete case whether the definition actually applies or not. A definition which does not stand this test is an artificial abstraction which has no place in mathematics. A definition of number must be such that the laws and fundamental operations of arithmetic can be naturally developed by means of it. This extremely practical tendency, as Kronecker himself called it, is one of the characteristic features of the volume.

The consideration of the nature of number and the fundamental operations is followed by a study of the divisibility of numbers, elementary congruences, the function $\varphi(m)$, and the theorems of Fermat and Wilson. The last lecture of this part is devoted to the invariants of a congruence. The function $f(x)$ is said to be an *invariant* of the congruence

$$k \equiv k' \bmod m$$

if it does not alter its value when x is replaced by any number which is congruent to it mod m. It is called a *characteristic* invariant if the congruence $k \equiv k' \bmod m$ is a direct consequence of the equation $f(k) = f(k')$. Only characteristic invariants are considered and it is observed that each of these invariants can be represented as a symmetric function of all the congruent numbers.

The second part begins with a study of congruences with respect to more than one modulus. If $m_1, m_2, \cdots, m_r$ are fixed numbers and if $c_1, c_2, \cdots, c_r$ assume all possible positive and negative integral values then the expression

$$c_1 m_1 + c_2 m_2 + \cdots + c_r m_r$$

will represent an infinite number of numbers. Each of these numbers is said to be divisible by the system $m_1, m_2, \cdots, m_r$. While the theory of these systems of divisors is practically superfluous when the numbers $m_1, m_2, \cdots, m_r$ are integers, it

will be found very useful in the domain of integral functions of one or more indeterminate numbers. The interesting generalization of the concept of multiplication of numbers which is furnished by the composition of modular systems and the method of finding the highest common divisor by reducing a modular system to its simplest equivalent system are striking instances of the extension of common processes.

The developments of the theory of domains of rationality are of especial interest since this useful concept was practically introduced into mathematics by Kronecker. One of the most important results of this theory is the resolution of the modular system $(p, x^{p^n} - x)$ into irreducible factors whose degrees are divisors of n. The greater part of the developments are with respect to domains of rationality of one variable, but the last lecture of this second part is devoted to the modular systems in the domain of more than one variable. An elementary exposition of some of the fundamental concepts of this part is given by Professor Fine in the article cited above.

By the application of analysis to problems in the theory of numbers, Dirichlet obtained some very important results for which purely arithmetic proofs are still wanting. Following the systematic arithmetic of Gauss, the work of Dirichlet gave the second principal direction to the investigations in number theory in the nineteenth century. In the third part of the present volume, analysis is employed to establish some of the earlier results as well as to deduce a number of new and very important properties.

Some of the arithmetic functions, such as the totient of a number, exhibit such great irregularities that an analytic representation seems hopeless. However, it is sometimes possible to find mean values of such functions which give a remarkably clear insight into their general behavior. Among the mean values considered are those of $\varphi(n)$, $\frac{\varphi(n)}{n}$, and the number of divisors of a number. One lecture is devoted to the fundamental properties of cyclotomic numbers and the function $x^n - 1$. The series of Dirichlet is studied to a considerable extent and its usefulness is illustrated by means of some examples.

The main subjects of the fourth and last part are the theory of power residues and the theorem announced by Legendre that every arithmetical progression whose first term and common difference are prime to each other represents an infinite number of prime numbers. Dirichlet gave the first rigorous proof of this famous theorem, but this proof

does not exhibit an interval in which a prime necessarily exists. In 1885 Kronecker succeeded in completing Dirichlet's proof in this important direction by showing that for each number m it is possible to determine a large number m' such that there is at least one prime, belonging to the arithmetic series in question, between m and m'. This instance serves to illustrate the high ideal which Kronecker set for himself and others in regard to completeness of proof.

It will be observed that the present volume does not enter upon the theory of forms, which occupies such a prominent place in the text-books on this subject. The elements of this theory are found in the works of Diophantus and his great commentator Fermat. At a later date Euler and Lagrange made a comprehensive study of the binary quadratic forms with respect to what numbers could be represented by them. Gauss was the first to study forms in general and to give a convenient notation for their further development. It is well known that a large part of the recent additions to number theory have been along this line.

Among the characteristic features of the present volume are the extensive use of congruences with respect to more than one modulus and the prominence of the concept of domain of rationality. Kronecker's work in this direction is so well known that it seems unnecessary to enter into details of explanations. It is perhaps sufficient to say that the editor has been remarkably successful in preserving the lecture style and thus he has given us a volume which forms unusually attractive reading, and, in places enters into details of criticism which are of especial interest to the beginner.

The extensive notes at the end of the volume greatly increase its value, as they give not only the sources from which the material has been gathered and the additions of the editor, but also corrections and references. The present volume was delayed by the fact that just before his death Kronecker was engaged upon a study of the decomposition of modular systems which was left unfinished. The editor has thought it wise to complete these investigations and incorporate them in these lectures. As this work has been completed it is hoped that the second volume will follow very soon and that the inspiration which Kronecker's lectures gave to the study of the theory of numbers in Germany may thus be spread still more widely and become more permanent.

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ERRATA.

THE following errata in the current volume of the BULLETIN have been brought to the notice of the editors:

Page 66, line 4, *for* $(x_i y_k z_{ik})$ *read* $(x_i x_k z_{ik})$; line 20, second sum, *for* $_{\lambda\rho}$ *read* $y_{\lambda\rho}$.

Page 68, ine 15, *for* y_{ii} *read* y_{ij}.

Page 69, line 32, *for* $(j = 1, \cdots, 6)$ *read* $(j, k = 1, \cdots, 6)$.

Page 263. In the announcement of the prize question of the Belgian academy of sciences, the last line of the page should read: n-linear forms, $n > 3$.

NOTES.

THE Librarian of the AMERICAN MATHEMATICAL SOCIETY acknowledges the gift to the library of twenty-one volumes of mathematical works presented by Messrs. Ginn & Co., eight by The Macmillan Company, three by Allyn and Bacon, five by the Open Court Publishing Company, two by Crane & Co. and fourteen by D. C. Heath & Co. Several members of the Society have also responded to the invitation to contribute their publications to the Society's library. The detailed list of bound volumes presented will appear in the Librarian's annual report.

THE second (April) number of Volume III. of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On the small divisors in the lunar theory," by E. W. BROWN; "On the holomorphisms of a group," by J. W. YOUNG; "A simple non-desarguesian plane geometry," by F. R. MOULTON; "On the real solutions of two linear homogeneous differential equations of the first order," by M. BÔCHER; "On a recent method for dealing with the intersections of plane curves," by C. A. SCOTT; "A complete set of postulates for the theory of absolute continuous magnitude," by E. V. HUNTINGTON; "Complete sets of postulates for the theories of positive integral and of positive rational numbers," by E. V. HUNTINGTON.

AT the meeting of the London mathematical society on March 13th, papers were read by Mr. G. H. HARDY on "The theory of Cauchy's principal values (III)," and by Rev. J. CULLEN on "The solutions of a system of linear congruences."

At the meeting of the Cambridge philosophical society on February 17th, Mr. G. H. Hardy read a paper on "The differentiation and integration of divergent series."

The members of the American Mathematical Society residing on the Pacific slope are arranging to hold a meeting early in May for the presentation and discussion of mathematical papers. Application has been made to the Council for the authorization of a Pacific Section.

The seventy-second annual meeting of the British association for the advancement of science will be held at Belfast, September 10–17. Section A (mathematical and physical science) will be presided over by Professor J. Purser, and Section G (engineering) by Professor J. Perry, F.R.S.

In connection with the publication of the complete works and correspondence of Christian Huyghens, Professor D. J. Korteweg, of Amsterdam, desires to examine a Latin pamphlet written about 1692 by a certain Hubertus Huyghens (Huigens or Hugenius), entitled: Animadversiones quaedam circa proportionem quam ad rectilineas habent figurae curvilineae. The pamphlet is needed to explain certain obscure references in the correspondence between Christian and Hubertus Huyghens, and cannot be found in the European libraries. If any reader of this note knows of the existence of the pamphlet in America, he will confer a favor by communicating the fact to the Librarian of this Society.

Professor Leo Königsberger, of the University of Heidelberg, has in preparation a biography of Professor Hermann von Helmholtz, which will shortly appear.

Cambridge University.—The Smith's prizes for 1902 have been awarded to T. H. Havelock for his essay "On the distribution of energy in the continuous spectrum," and to J. E. Wright for his essay on "Singular solutions of differential equations with known infinitesimal transformations." H. E. Wimperis received a mention for his essay on "The temperature of meteorites."

Oxford University.—The following mathematical lectures are announced for the Easter and Trinity terms, 1902: —By Professor A. E. H. Love: Waves and sound, three hours.—By Professor E. B. Elliott: Elements of the theory of functions, three hours.—By Professor W. Esson: Synthetic geometry of conics. one hour; Informal instruction, one hour.—By Mr. J. E. Campbell: Some con-

tact transformations, one hour.—By Mr. A. L. DIXON: Calculus of variations, one hour.—By Mr. H. T. GERRANS: Line geometry, two hours.—Mr. A. E. JOLLIFFE: Higher analytical plane geometry, two hours; Series, one hour.—By Mr. P. J. KIRKBY: Higher plane curves, two hours.—By Mr. E. H. HAYES: Attractions. two hours.—By Mr. J. W. RUSSELL: An elementary course of rigid dynamics, three hours.—By Mr. C. LEUDESDORF: Determinants, two hours first half of term.—By Mr. A. L. PEDDER: Spherical trigonometry, one hour.—By Mr. C. E. HASELFOOT: Theory of equations, one hour.—By Mr. R. F. MCNEILE: Elementary integral calculus, two hours.—By Mr. C. H. THOMPSON: Differential equations, two hours.

UNIVERSITY OF CALIFORNIA.—The following courses in mathematics, mathematical physics and astronomy are announced for the academic year 1902–1903:—By Professor I. STRINGHAM: Analytic projective geometry, three hours; Logic of mathematics, two hours, first semester; History of mathematics, three hours, second semester; seminar, two hours.—By Professor G. C. EDWARDS: Differential equations, three hours; Theory of equations, two hours.—By Dr. E. J. WILCZYNSKI: Linear differential equations, three hours; Analytic geometry of space, three hours, second semester.—By Mr. A. W. WHITNEY: Theory of probabilities, two hours, first semester; Theory of functions of a complex variable, three hours, first semester.—By Dr. C. A. NOBLE: Advanced calculus, three hours; Elliptic functions, three hours, second semester.—By Dr. D. N. LEHMER: Synthetic geometry, three hours, second semester.—By Dr. E. M. BLAKE: Differential geometry, three hours.—By Dr. T. M. PUTNAM: Theory of groups, three hours, first semester.—By Professor KOWER: Descriptive geometry, six hours; Graphostatics, two hours, first semester.—By Professor F. SLATE: Analytic mechanics, four hours; Dynamics of rotation, two hours, second semester.—By Professor RAYMOND: Harmonic motion, two hours, second semester.—By Professor HESSE: Hydrodynamics, three hours, first semester.—By Mr. LECONTE: Kinematics, two hours.—By Mr. ————: Theory of light, two hours.—By Dr. BURGESS: Energetics, two hours, first semester.—By Professor LEUSCHNER: Theoretical astronomy, two hours; Perturbations and celestial mechanics, four hours.—By Dr. TOWNLEY: Advanced practical astronomy, three hours, first semester; Measurement and reduction of astronomical photographs and spectrograms, two hours, first semester; Mechanical quadratures, three hours, second semester.

During the summer session, June 26–August 6, 1902, Dr. E. J. WILCZYNSKI will give a course on the theory of functions, five hours per week.

UNIVERSITY OF MICHIGAN.—The following courses in mathematics are announced for the academic year 1902–1903:—By Professor W. W. BEMAN: Solid analytic geometry, two hours, first semester; Differential equations, three hours, first semester; Advanced differential and integral calculus, two hours; Higher plane curves, two hours, second semester; Linear differential equations, two hours, second semester; Quaternions, two hours, second semester. —By Professor A. ZIWET: Advanced mechanics, three hours, second semester; Theory of the potential, three hours, first semester.—By Professor J. L. MARKLEY: Projective geometry and modern analytic geometry, three hours; Theory of functions, three hours.—By Dr. J. W. GLOVER: Higher algebra, three hours; Theory of substitutions, two hours.—By Mr. E. B. ESCOTT: Theory of numbers, two hours, second semester.

STANFORD UNIVERSITY.—The following courses in pure mathematics are announced for the academic year 1902–1903:—By Professor R. E. ALLARDICE: Definite integrals, two hours; Invariants, two hours; Geometry of three dimensions, three hours.—By Professor R. L. GREEN: Theory of equations, three hours.—By Professor H. F. BLICHFELDT: Non-euclidean geometry, two hours.—By Professor G. A. MILLER: Theory of groups, three hours; Theory of numbers, two hours; Seminary in the theory of groups, two hours.

YALE UNIVERSITY.—The following mathematical courses are offered next year:—By Professor J. W. GIBBS: Elementary vector analysis (first semester); Advanced vector analysis (second semester), three hours; Multiple algebra, two hours; Electromagnetic theory of light, two hours.—By Professor W. A. BEEBE: Elementary analytical mechanics (first semester); Celestial mechanics (second semester), three hours.—By Professor JAMES PIERPONT: Advanced calculus, three hours; Projective geometry, three hours; Advanced theory of functions of a complex variable, three hours.—By Professor P. F. SMITH: Advanced analytical geometry, two hours, first semester, and three hours, second semester; Differential equations, two hours (first semester). —By Professor H. A. BUMSTEAD: Problems in mathematical physics, two hours.—By Professor M. B. PORTER: Differential equations and theory of functions (first semes-

ter), and Invariants (second semester) three hours.—By Dr. H. E. HAWKES: Higher algebra, three hours; Complex multiplication of elliptic functions (second semester), three hours.—By Dr. W. A. GRANVILLE: Advanced analytic mechanics, two hours.—By Dr. E. R. HEDRICK: Partial differential equations, three hours.—By Dr. A. S. GALE: Elementary analytic geometry (second course), three hours; Theory of transformations of space (first semester), three hours.

THE several universities below offer during the summer semester of the current academic year courses in mathematics as follows:

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Seminary exercises in definite integrals, two hours; Plane analytic geometry, four hours; Modern methods in analytic geometry, two hours.—By Professor R. STURM: Seminary exercises, two hours; Geometrical chapters of mechanics, three hours; Elements of line geometry, three hours.—By Dr. F. LONDON: Theory of elliptic functions, four hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN: Differential equations, four hours; Algebra, four hours; Seminar, three hours.—By Professor M. NÖTHER: Synthetic geometry with exercises, three hours; Differential geometry of curves and surfaces, three hours; Introduction to elliptic functions, three hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Algebra, four hours; Selected chapters of analytic geometry, four hours; Seminar, one hour.—By Professor E. NETTO: Plane analytic geometry, four hours; Elliptic functions, four hours; Seminar, one hour.—By Professor R. HAUSSNER: Integral calculus, three hours; Theory of probability, two hours; Descriptive geometry, five hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Higher mechanics, four hours; Seminar, two hours.—By Professor D. HILBERT: Differential and integral calculus, four hours; Foundations of geometry, two hours; Selected chapters of the theory of potential, two hours; Seminar in the theory of functions, two hours.—By Professor M. BRENDEL: Selected chapters of higher analysis, three hours, with exercises, two hours; Gyldén's theory of perturbations, three hours.—By Professor F. SCHILLING: Analytic geometry, four hours; Perspective, one hour, with exercises,

two hours; Seminar, exercises in geometry, one hour.—By Professor K. BOHLMANN: Social insurance, two hours, with seminar exercises, two hours.—By Dr. E. ZERMELO: Calculus of variations, two hours; Exercises in differential and integral calculus, two hours; Exercises in higher analysis, two hours.—By Dr. O. BLUMENTHAL: Galois's theory of algebraic equations, three hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Elliptic functions, II, four hours; Selected chapters in the theory of analytic functions, one hour; Seminar, one hour.—By Professor E. STUDY: Mechanics I, four hours; Selected chapters in the theory of differential equations, two hours; Seminar.—By Professor G. KOWALEWSKI: Integral calculus, four hours, with exercises, one hour; Introduction to differential geometry, two hours; Repetitorium of analytic geometry, two hours.

UNIVERSITY OF HEIDELBERG.—By Professor L. KÖNIGSBERGER: Differential and integral calculus, four hours; Theory of curves and surfaces. four hours; Seminar, two hours.—By Professor M. CANTOR: Plane analytic geometry. four hours; Arithmetic and algebra, three hours.—By Professor F. EISENLOHR; Theory of probability, three hours; Mechanics, four hours.—By Professor K. KOEHLER: Plane synthetic geometry, three hours.—By Professor G. LANDSBERG: Theory of determinants, two hours; Theory of numbers, two hours.—By Dr. K. BOEHM: Selected chapters of higher mechanics, one hour.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Elliptic functions, four hours, with applications, one hour; Projective geometry, two hours.—By Professor A. GUTZMER: Plane analytic geometry, four hours; Calculus of variations, three hours; Historical development of analysis, one hour.—By Professor G. FREGE: Differential and integral calculus with exercises, five hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Theory of curved lines and surfaces, four hours: General theory of functions, four hours; Seminar, two hours.—By Professor E. HESS: Differential calculus, five hours; Selected chapters of geometry, three hours; Seminar. three hours.—By Dr. F. v. DALGWICK: Theory of conics, four hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Introduction to synthetic geometry, two hours; Technical

mechanics, four hours; Seminar, two hours.—By Professor H. WEBER: Calculus of variations, four hours; Encyclopædia of elementary mathematics, two hours; Seminar, two hours.—By Professor F. ROTH: Differential and integral calculus, three hours, with exercises; Plane analytic geometry, three hours.—By Professor A. KRAZER: Definite integrals, three hours; Introduction to the theory of functions, three hours; Determinants, two hours; Seminar, one hour.—By Dr. E. TIMERDING: Theory of tides, two hours.—By Dr. J. WELLSTEIN: Selected chapters of applied mathematics, two hours: Arithmetic theory of algebraic functions, two hours.

UNIVERSITY OF TÜBINGEN.—By Professor A. v. BRILL: Analytic mechanics, five hours; Selected chapters of the theory of curves and surfaces, two hours; Seminar, two hours.—By Professor H. STAHL: Elementary analysis, three hours; Theory of functions, three hours; Seminar, two hours.—By Professor L. MAURER: Higher analysis, four hours, with exercises, two hours; Invariant theory of binary forms, two hours.

THE following noteworthy catalogues of mathematical books have recently appeared: Lagerverzeichniss 201 der Buchhandlung Gustav Fock, Schlossgasse 7, Leipzig; 1096 numbers.—H. Welter, 4 Rue Bernard-Palissy, Paris, catalogue no. 112; 282 numbers on algebra and trigonometry, 215 sets of journals.

PROFESSOR RICHARD DEDEKIND celebrated on March 18 the fiftieth anniversary of his doctorate. On this occasion, the University of Strassburg conferred upon him an honorary doctor's degree, and a congratulatory address was presented by the Deutsche Mathematiker-Vereinigung. In accordance with a German custom, his doctor's diploma was renewed by the University of Göttingen. Professor Dedekind has been connected with the Technical high school at Brunswick since 1862, and still delivers lectures, although he has been nominally retired for some years.

AT Yale University the following changes have taken place in the mathematical faculty: Dr. M. B. PORTER has been promoted from the position of instructor to that of assistant professor. Dr. H. E. HAWKES, who has been studying in Göttingen during the past year, has recently recovered from a severe illness and will resume his work as instructor in the fall. Dr. E. B. WILSON has been granted a year's leave of absence for study abroad.

PROFESSOR M. W. HASKELL, of the University of California, has received leave of absence and will spend the coming year abroad.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALEKSEIEF (V. G.). Theory of the rational invariants of binary forms according to Sophus Lie, Cayley, and Aronhold. Dorpat, 1899. 8vo. 8+232 pp. (Russian.)

——. Theory of rectilinear congruences in connection with the theory of surfaces, according to Prof. Darboux's lectures. Moscow, 1899. 8vo. 98 pp. (Russian.)

ALTMEYER (A.). Ueber Tetraeder mit Höhenschnittpunkt bei einer Fläche zweiter Ordnung. (Diss.) Strassburg, 1901. 8vo. 47 pp.

BASSET (A. B.). An elementary treatise on cubic and quartic curves. London and New York, Macmillan, 1901. 8vo. 16+255 pp. Cloth. (Cambridge University Press.) 10s. 6d.

BLACK (C. W. M.). The parametric representation of the neighborhood of a singular point of an analytic surface. (Diss., Harvard University.) 8vo. 1902. (*Proceedings of the American Academy of Arts and Sciences* 37, pp. 281–330.)

DUPORCQ (E.). Procès-verbaux sommaires du deuxième congrès international des mathématiciens, tenu à Paris du 6 au 12 août 1900. Paris, Imprimerie nationale, 1901. 8vo. 17 pp. (Exposition universelle internationale de 1900; ministère du commerce.)

EGOROF (D. F.). Partial differential equations of the second order with two independent variables; general theory of the integrals; characteristics. Moscow, 1898. 8vo. 392 pp. (Russian.)

EMCH (A.). Cyclographic transformation of ordinary space. 8vo. 1901. (*The University of Colorado Studies* 1, pp. 33–43.)

——. On the congruences of twisted curves. 8vo. 1900. (*The University of Colorado Studies* 1, pp. 29–32.)

ERMAKOF (V. P.). Differential calculus. Lectures delivered at the Kief Polytechnic Institute. Part I: Differentiation of functions; Part II: Investigation of functions; Part III: Applications to geometry. Kief, 1899. 8vo. 52+100 pp. (Russian.)

FUNK (R.). Die Konfiguration $(16_4, 20_5)$, ihre analytische Darstellung und ihre Beziehungen zu gewissen algebraischen Flächen. (Diss.) Strassburg, 1901. 8vo. 32 pp.

GAUSS (K. F.). General investigations of curved surfaces, of 1827 and 1825. Translated, with notes and a bibliography, by J. C. Morehead and A. M. Hiltebeitel. Princeton, The University Library, 1902. 4to. 8+127 pp. Cloth. $1.75

GRACE (J. H.) and ROSENBERG (F.). The elements of coordinate geometry. Part 2: The conics. 2d edition. London, Clive, 1902. 8vo. 338 pp. 4s. 6d.

GRADHAUDT (E.). Beiträge zur Theorie der Fokaleigenschaften der Krümmungskurven auf den Flächen zweiter Ordnung. (Diss.) Rostock, 1901. 8vo. 58 pp.

HILTEBEITEL (A. M.). See GAUSS (K. F.).

JAHRBUCH über die Fortschritte der Mathematik. Herausgegeben von E. Lampe und G. Wallenberg. Vol. 30: 1899. Heft 3. Berlin, Reimer, 1901. 8vo. 65 pp. and pp. 609–918. M. 12.00

LAISANT (C. A.). Interprétation géométrique des dérivées partielles dans la théorie des courbes et des surfaces algébriques. 8vo. 14 pp. (*L'enseignement mathématique* 3, novembre 1901.)

LAMPE (E.). See JAHRBUCH.

MOREHEAD (J. C.). See GAUSS (K. F.).

ROSENBERG (F.). See GRACE (J. H.).

ROSHKOF (D. P.). See TISSOT (M. A.).

ROSSI (O.). Sui gruppi tre volte transitivi, di ordine minimo compatibile col loro grado di transitività; nota. [Salerno] Jovane, 1901. 8vo. 12 pp.

SCHEFFERS (G.). Anwendung der Differential- und Integralrechnung auf Geometrie. Vol. II: Einführung in die Theorie der Flächen. Leipzig, Veit, 1902. 8vo. 10+518 pp. Cloth. M. 13.00

SONNET (H.). Premiers éléments du calcul infinitésimal, à l'usage des jeunes gens qui se destinent à la carrière d'ingénieur. Paris, Hachette, 1902. 8vo. 3+360 pp. Fr. 6.00

TIKHOMANDRITSKY (M.). Lectures on the theory of probabilities. Kharkof, University Press, 1898. 8vo. 7+103 pp. (Russian.)

TISSOT (M. A.). The representation of one surface on another and the construction of maps. Translated by D. P. Roshkof. Moscow, 1899. 8vo. 298 pp. (Russian.) R. 3.00

WALLENBERG (G.). See JAHRBUCH.

II. ELEMENTARY MATHEMATICS.

ALEY (R. J.). Graphs. Boston, Heath, 1902. 16mo. 4+21 pp. (Heath's mathematical monographs, No. 6.) $0.10

AMATO (F. B.). Costruzione e dimostrazione per la trisezione di qualunque angolo. Mistretta, Tipografia del *Progresso*, 1901. 8vo. 7 pp.

BARDEY (E). Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, vorzugsweise für Realschulen, höhere Bürgerschulen und verwandte Anstalten neu bearbeitet von H. Hartenstein. Ausgabe B ohne Logarithmentafel. 4te Auflage. Leipzig, Teubner, 1902. 8vo. 4+170 pp. Cloth. M. 2.00

BOLTE (F.). Leitfaden für den Unterricht in der Arithmetik, zum Gebrauche an Navigationsschulen bearbeitet. 3te Auflage. Hamburg, Peuser, 1902. 8vo. 79 pp. Boards. M. 2.00

——. Leitfaden für den Unterricht in der Planimetrie, mit einem Anhange über Körperberechnungen, zum Gebrauche an Navigationsschulen bearbeitet. 3te Auflage. Hamburg, Peuser, 1902. 8vo. 56 pp. Boards. M. 1.50

——. Leitfaden für den Unterricht in der Stereometrie und sphärischen Trigonometrie zum Gebrauche an Navigationsschulen bearbeitet. Hamburg, Peuser, 1902. 8vo. 37 pp. Boards. M. 1.00

BUFFA (P.). Primo studio della geometria piana per le scuole secondarie inferiori. Torino, Paravia, 1901. 8vo. 12+154 pp. Fr. 2.00

CALCUL mental, par un ancien élève de l'Ecole polytecnique. Paris, Tricon, 1901. 16mo. 74 pp. (Cours complet d'arithmétique théorique et pratique.)

DIEKMANN (J.). See KOPPE (K.).

ELEMENTOS de aritmetica, con algunas nociones de algebra, por los Hermanos de las escuelas cristianas. 5a edición, correspondiente á los cursos medio y superior. Paris, 1902. 16mo. 392 pp.

FRASCHIGNI (E.). Le rette parallele. Rifacimento, con dichiarizioni e aggiunte, di uno studio già pubblicato col titolo "La geometria immaginaria." Bologna, Zanichelli, 1901. 8vo. 31 pp. Fr. 2.00

GAUSS (F. G.). Fünfstellige vollständige trigonometrische und polygonometrische Tafeln für Maschinenrechnen. Teilung des Quadranten in 90 Grade zu 60 Minuten. Halle, Strien, 1901. 8vo. 100+18 pp. Cloth. M. 7.00

GREFE (A.). Fünfstellige logarithmische und trigonometrische Tafeln, nebst einer gröaseren Anzahl von Hilfstafeln. 9te Auflage. Bielefeld, 1901. 8vo. 4+180 pp., 1 plate. Cloth. M. 2.00

GUNDELFINGER (S.). Sechsstellige Gaussische und siebenstellige gemeine Logarithmen. 2te, durch eine Ergänzungstabelle vermehrte Ausgabe. Leipzig, Veit, 1902 4to. 4 + 31 pp. Cloth. M. 2.80

GUYOU (E.). De l'extension du système décimal à la mesure de la circonférence. Paris, 1901.

HAMMER (E.). Sechsstellige Tafel der Werte $\log_{10}\frac{1+x}{1-x}$ für jeden Wert des Arguments log x von 3.0–10 bis 9.99000–10. (Vom Argument 9.99000–10 an bis 9.999700–10 sind die log $\frac{1+x}{1-x}$ nur noch *fünfstellig* angegeben, von dort an *vierstellig*.) Leipzig, Teubner, 1902. 8vo. 4+73 pp. Boards. M. 3.60

HARTENSTEIN (H.). See BARDEY (E.).

HARTL (H.). Aufgaben aus der Arithmetik und Algebra. 2te Auflage. Resultate. Wien, Deuticke, 1902. 8vo. 3 + 96 pp. M. 1.50

HOČEVAR (F.). Arithmetik und Algebra, nebst einer Sammlung von Uebungsaufgaben für Obergymnasien. Wien und Prag, Tempsky, 1902. 8vo. 3+261 pp. Cloth. M. 3.60

INGRAMI (G.). Aritmetica generale ed algebra ad uso del liceo. 2a edizione rifatta. Bologna, Cenerelli, 1901. 8vo. 144 pp. Fr. 2.50

KOPPE (K.). Geometrie zum Gebrauche an höheren Unterrichtsanstalten vollständig neu bearbeitet von J. Diekmann. 19te Auflage (3te Auflage der neuen Bearbeitung), nebst zahlreichen Uebungen und Aufgaben. Teil 1: Planimetrie. Ausgabe für Gymnasien. Essen, Baedeker, 1902. 8vo. 6+208 pp. Cloth. M. 2.40

KRÜGER (R.). Ebene Trigonometrie. Unterweisungen und Aufgaben. 6te Auflage. Strelitz, Hittenkofer, 1902. 8vo. 27 pp. (Methode Hittenkofer, No. 57.) M. 2.00

LÜBECK (O.). Stereometrie. Unterweisungen und Aufgaben. 5te Auflage. Strelitz, Hittenkofer, 1902. 8vo. 42 pp. (Methode Hittenkofer, No. 58.) M. 2.40

MANTOVANI (E.). Diario aritmetico per la classe quinta elementare. Milano, Vallardi, 1901. 16mo. 70 pp. Fr. 0.50

MARCOU (C. A.). Oxford questions in arithmetic and algebra. Papers set in responsions from Hilary term 1896 to September, 1901. With answers. London, Simpkin, 1902. 12mo. 3s. 6d.

MARTUSCELLI (M.). Numeri irrazionali e limiti, ad uso dei licei. Salerno, Jovane, 1901. 8vo. 17 pp. Fr. 0.60

PASQUALI (P.). Geometria intuitiva ad uso delle scuole tecniche. normali, industriali. Lezioni di ritaglio geometrico conforme ai programmi governativi. Vol. II. Parma, Battei, 1901. 16mo. 67 pp. Fr. 0.60

PIFFL (H.). Aufgabensammlung aus der Algebra. Auflösungen. Wien, Stock. 1901. 8vo. 64 pp. M. 0.50

PROELL (R.). Rechentafel für Multiplikation, Division, Potenzierung und Radizierung. Dresden, 1901. 12mo. 13 pp., 2 plates. M. 2.00

SCHULTZ (E.). Leitfaden der Planimetrie für gewerbliche Lehranstalten. Teil I. 3te Auflage. Essen, Baedeker, 1902. 8vo. 4+82 pp. Cloth. M. 1.00

SCHWARZ (H.). Algebra. Teil 2. Unterweisungen und Aufgaben. 6te Auflage. Strelitz, Hittenkofer, 1902. 8vo. 31 pp. (Methode Hittenkofer, No. 56.) M. 2.00

SCOTTI (G.). Aritmetica pratica ad uso del ginnasio inferiore e dei corsi complementari, secondo gli ultimi programmi governativi. 12a edizione accuratamente riveduta ed ampliata dall'autore. Torino, Tipografia Salesiana, 1902. 16mo. 259 pp. Fr. 1.00

SEVERA (V.). Elementi di trigonometria piana ad uso dei licei e degli istituti tecnici, compilati conformamente agli ultimi programmi governativi. Pitigliano, Paggi, 1902. 8vo. 76 pp., 2 plates.

TABLES des logarithmes à cinq décimales des nombres naturels de 1 à 10,000, et tables des logarithmes à cinq décimales des sinus, cosinus, tangentes et cotangentes des arcs du premier quadrant, de minute en minute centésimale, dans le système de la division décimale de la circonférence. Paris, Delagrave, 1901. 18mo. 286 pp.